Part B: ENGINEERING ECONOMY

ANNUITIES

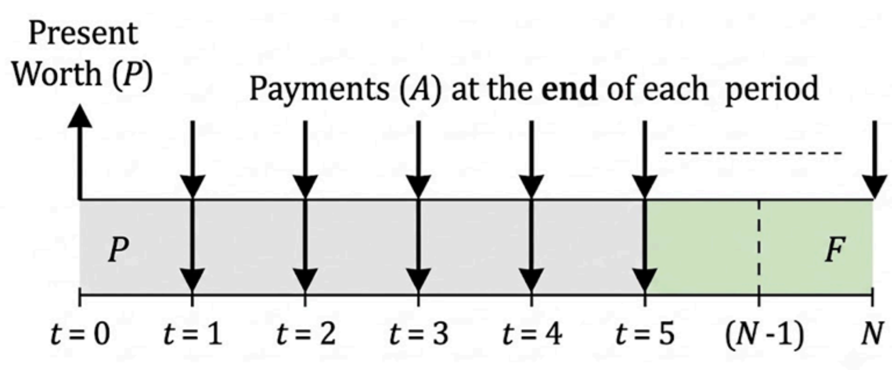
An **annuity** is defined as a series of equal payments (or receipts) made at equal intervals of time. In engineering economic analysis, annuities model structured financial transactions where capital is accumulated or systematically paid off over a specified duration. Common real-world applications of annuities include paying off loan amortizations, funding insurance premiums, distributing scholarship programs, establishing sinking funds to replace depreciable capital assets, and managing long-term time deposits.

To evaluate the economic viability of these cash flows, financial mathematics establishes specific mathematical relationships among the periodic payment (A), the interest rate per period (i), the total number of compounding periods (N), the present worth (P), and the future worth (F). Depending on when the individual payments occur relative to the interest-accruing periods, annuities are classified into four distinct categories: ordinary annuity, annuity due, deferred annuity, and perpetuity.

Four Types of Annuities:

1. Ordinary Annuity

An **ordinary annuity** represents the foundational model of uniform cash flows. It is a type of annuity where the periodic payments are made at the **end of each period**. Because the initial payment occurs at the end of the first period (at time t = 1), it does not accrue interest during that first compounding interval. The present worth (P) of an ordinary annuity is evaluated exactly one period before the first payment occurs (t = 0), whereas its future worth (F) is evaluated exactly at the moment the final payment is made (t = N).



P payments are made at the **END** of each period.

To determine the present worth (P) of an ordinary annuity, we discount each individual payment back to the present time and sum the resulting geometric progression. This yields the standard present value equation:

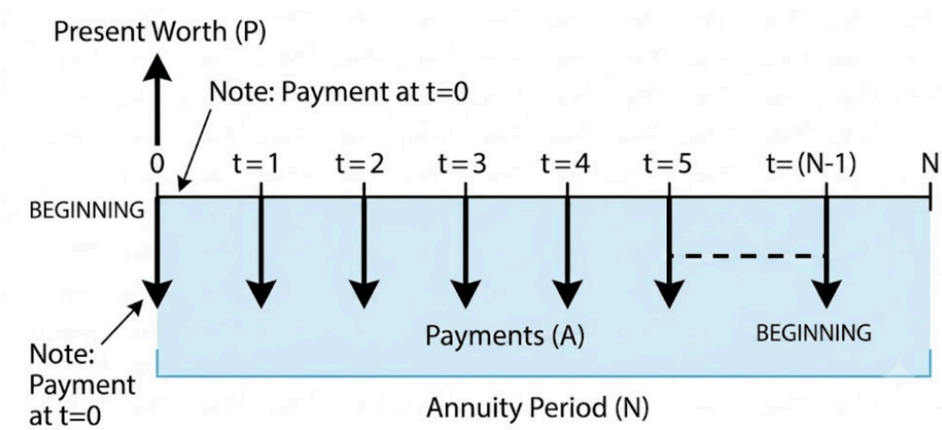
$$P = \frac{A}{i}\left[1 - (1 + i)^{-N}\right]$$

To find the accumulated future worth (F) at the end of the N-th period, each payment is compounded forward to the final time interval. Summing this progression provides the standard compound amount factor equation:

$$F = \frac{A}{i}\left[(1 + i)^{N} - 1\right]$$

2. Annuity Due

An **annuity due** is a type of annuity where the uniform payments are made at the **beginning of each period**. The primary distinction between an ordinary annuity and an annuity due lies in the timing of the cash flows: because each payment occurs at the start of a period (beginning at t = 0), every single cash flow earns an additional period of interest compared to an ordinary annuity.

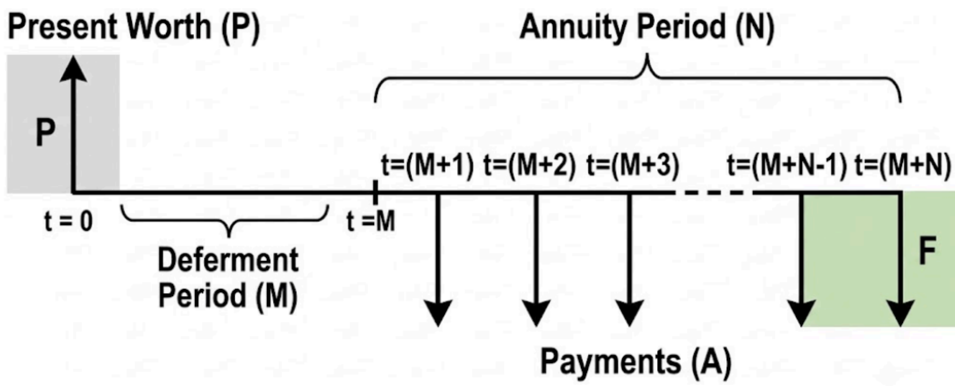


When deriving the formula for the present worth (P) of an annuity due, we can treat the very first payment at t = 0 as a standalone cash flow that requires no discounting. The remaining (N - 1) payments then behave exactly like an ordinary annuity spanning (N - 1) periods. By mathematically combining the immediate initial payment with the discounted remaining payments, we arrive at the corrected and complete present worth formula:

$$P = A + \frac{A}{i}\left[1 - (1 + i)^{-(N-1)}\right]$$

2. Deferred Annuity

A **deferred annuity** is a structured type of annuity where the sequence of uniform periodic payments does not begin immediately, but is instead **deferred for several periods of time**. In these cash flow structures, the financial timeline is divided into two distinct phases: the *period of deferment* (let M be the number of deferred interest periods where no payments occur) and the *finitude of the annuity* (let N be the total number of actual payments made).



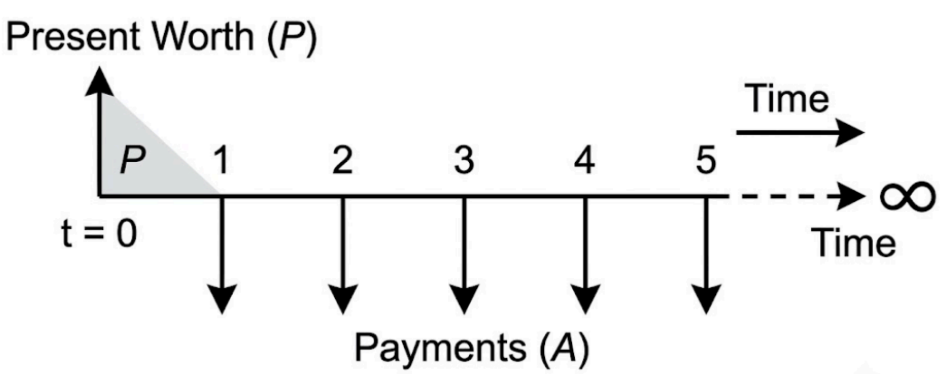
The ordinary annuity (N payments) is deferred for M periods.

To evaluate the present worth (P) of a deferred annuity at the true origin (t = 0), the cash flow is analyzed in two steps. First, the N uniform payments are treated as an ordinary annuity to find their equivalent value (P') exactly one period before the first payment begins (which lands at the end of the deferment period, t = M). Second, this intermediate single sum P' must be discounted back across the M silent periods to the true present timeline (t = 0) using the single-payment present worth factor $(1 + i)^{-M}$. Combining these operations yields the comprehensive formulation:

$$P = \frac{A}{i}\left[1 - (1 + i)^{-N}\right](1 + i)^{-M}$$

4. Perpetuity

A **perpetuity** is a special classification of annuity that follows the same structural mechanics as an ordinary annuity, with the unique exception that the uniform **payments continue infinitely** ($N \to \infty$). Because the cash flows never cease, a perpetuity can never accumulate a terminal "future worth" (F = ∞). However, it possesses a finite and calculable present worth (P), which represents the capitalized cost or the principal sum required today to yield an infinite stream of periodic payouts without ever depleting the core fund.



Payments (A) continue infinitely (N → ∞)

Mathematically, as the number of periods (N) approaches infinity, the discount component $(1 + i)^{-N}$ within the ordinary annuity equation diminishes and approaches zero $\left(lim_{N \to \infty} (1 + i)^{-N} = 0\right)$. Substituting this limit into the standard present value formula simplifies the equation down to a basic, elegant ratio of the periodic payment to the interest rate:

$$P = \frac{A}{i}$$