Fundamentals of Algebra

PROPERTIES OF ALGEBRA

1. **Commutative Properties**
   These properties state that changing the order of the numbers does not change the final result.

   a. Commutative Property of Addition
   When you are adding numbers, it doesn't matter which one comes first.

   The Rule: $a + b = b + a$

   Example: $2x + 3y$ is exactly the same as $3y + 2x$

   b. Commutative Property of Multiplication
   When you are multiplying numbers, the order does not change the product.

   The Rule: $ab = ba$

   Example: $(4 - x)y^2$ gives the same result as $y^2(4 - x)$

   Note: The commutative property **only** works for addition and multiplication. It does *not* work for subtraction $(5 - 3 \neq 3 - 5)$ and division $(6 \div 2 \neq 2 \div 6)$

2. **Associative Properties**
   These properties state that changing the grouping of the numbers (using parentheses) does not change the final result.

   a. Associative Property of Addition
   When adding three or more terms, you can group them however you like.

   The Rule: $(a + b) + c = a + (b + c)$

   b. Associative Property of Multiplication
   When multiplying three or more terms, shifting the parentheses changes nothing about the final product.

   The Rule: $(a\,b)c = a(b\,c)$

3. **Distributive Properties**
   If a term is multiplied by a sum inside parentheses, you must multiply that outside term by every single term inside the parentheses.

   The Rule: $a(b + c) = ab + ac$ or $(a + b)c = ac + bc$

4. **Identity Properties**
   The identity properties look for a number that, when applied, leaves the original value completely unchanged.

   a. Additive Identity Property
   Adding zero to any number keeps it exactly the same. Zero is the "additive identity."

   The Rule: $a + 0 = a$

   b. Multiplicative Identity Property
   Multiplying any number by 1 keeps it exactly the same. One is the "multiplicative identity."

   The Rule: $a \cdot 1 = a$

5. **Inverse Properties**
   Inverse properties look for the number you can apply to a value to return to the identity element (0 for addition,1 for multiplication).

   a. Additive Inverse Property
   If you add a number to its exact opposite (its negative), they cancel each other out to equal 0. This opposite is called the additive inverse.

   The Rule: $a + (-a) = 0$

   b. Multiplicative Inverse Property (Reciprocals)
   If you multiply a number by its reciprocal (flipped fraction version), the result is always 1.

   The Rule: $a \cdot \frac{1}{a} = 1 \qquad (where\ a \neq 0)$

   Why can't a=0? Because division by zero (01) is undefined in mathematics. Therefore, zero is the only number that does not have a multiplicative inverse.

LAWS OF EXPONENTS

An exponent (or power) tells us how many times a base number is multiplied by itself. For example, in x^3, x is the base and 3 is the exponent, meaning $x \cdot x \cdot x$.

1. **Rules of 1**

   Any number raised to the power of 1 is simply equal to itself. It means the base appears exactly one tim

   The Rule: $x^1 = x$

2. **The Product Rule**

   When multiplying exponential terms that have the same base, keep the base the same and add the exponents together.

   The Rule: $x^m \cdot x^n = x^{m+n}$

   Example: $3^2 \cdot 3^5 = 3^{2+5} = 3^7$

3. **The Power Rule**

   When raising a power to another power, keep the base the same and multiply the exponents together.

   The Rule: $\left(x^m\right)^n = x^{m \cdot n}$

   Example: $\left(8^2\right)^3 = 8^{2 \cdot 3} = 8^6$

4. **The Quotient Rule**

   When dividing exponential terms with the same base, keep the base the same and subtract the exponent in the denominator (bottom) from the exponent in the numerator (top).

   The Rule: $\frac{x^m}{x^n} = x^{m-n} \quad (where\ x \neq 0)$

5. **The Zero Rule**

   Any non-zero number raised to the power of zero is always exactly 1.

   The Rule: $x^0 = 1$

6. **Negative Exponents**
   A negative exponent tells you that the term belongs on the opposite side of a fraction. To make the exponent positive, flip the term into the denominator as a reciprocal.

   The Rule: $x^{-n} = \frac{1}{x^n}$

LAWS OF RADICALS: FRACTIONAL EXPONENTS

Radicals (roots) and exponents are two sides of the same coin. Any radical expression can be rewritten as an expression with a fractional (rational) exponent.

Conversion Rule: $a^{\frac{x}{y}} = \sqrt[y]{a^x}$

When converting a fractional exponent to a radical, the **numerator (x) stays as the power**, and the **denominator (y) becomes the index** (the root value).

In $\sqrt[y]{a^x}$, the symbol $\sqrt{\ }$ is the radical, a^x is the radicand, and y is the index. If no index is written (e.g., $\sqrt{x}$), it is understood to be a square root (index of 2).

THE CONCEPT OF INFINITY (∞)

Infinity is not a real number; rather, it is a conceptual tool representing a quantity that grows endlessly without bound.

$$\frac{a}{0} = \pm\ \infty \qquad (where\ a \neq 0)$$

If you divide a constant number by a value that approaches zero, the result becomes infinitely large.

The Inverse Property: *"the inverse of a small number is a large number."* For example, $\frac{1}{0.1} = 10$, $\frac{1}{0.01} = 1000$. As the denominator shrinks to infinitely close to 0, the total value explodes toward infinity.

Arithmetic Rules with Infinity

Because infinity is a concept and not a finite number, it does not obey ordinary arithmetic rules. It absorbs finite numbers:

- **Addition:** $\infty + 2 = \infty$ (Adding a tiny drop to an infinite ocean leaves it infinite).
- **Subtraction:** $\infty - 3 = \infty$ (Taking away a finite amount leaves it infinite).
- **Multiplication:** $\infty \cdot 2 = \infty$ (Doubling infinity is still just infinity).

INDETERMINATE FORMS

An expression is called **indeterminate** if it cannot be precisely determined through basic arithmetic rules. These forms usually appear in calculus when evaluating limits, signaling that more algebraic work is required to find a definitive answer.

	Form	Why it is Indeterminate
Quotients	$\frac{0}{0}$	Division by zero implies infinity, but zero divided by anything implies zero. These two rules conflict.
	$\frac{\infty}{\infty}$	Does the numerator grow faster, or does the denominator? The balance determines the answer.
Products	$0 \cdot \infty$	Zero multiplied by anything is zero, but infinity multiplied by anything is infinity.
Differences	$\infty - \infty$	This is not 0. We do not know which infinity is "bigger" or growing faster without more context.
Powers	0^o	Any number to the power of 0 is 1, but 0 to any power is 0.
	1^∞	1 multiplied by itself any number of times is 1, but an exponent approaching infinity alters behavior.
	∞^0	An infinite base wants to explode, but a zero exponent wants to force the value to 1.

When you encounter an indeterminate form, it does not mean "no answer." It means the true value is hidden, and you must use tools like algebraic factoring or calculus techniques (such as L'Hôpital's Rule) to solve it

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FACTORING TECHNIQUES

Factoring is the mathematical process of breaking down a polynomial expression into a product of simpler terms or "factors." It is essentially the reverse process of multiplication or expansion.

1. **Common Factor (Greatest Common Factor - GCF)**
   Look for the highest number and variable power that divides evenly into all terms.

$$3a^2b^2 - 6a^2b + 6a$$

Factoring $3a^2$ leaves:

$$3a^2(b^2 - 2b + 2)$$

2. **Difference of Squares**
   Take the square root of the first term and the square root of the second term. Write them twice: once with a minus sign, and once with a plus sign.

   The Formula: $a^2 - b^2 = (a - b)(a + b)$

   Note: A *sum* of squares ($a^2 + b^2$) cannot be factored using real numbers.

3. **Difference of Cubes**
   Like squares, two perfect cubes being subtracted follow a highly predictable pattern.

   The Formula: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

   Memory Trick (SOAP): The signs in the factored form always follow the acronym **SOAP**: **S**ame sign (-), **O**pposite sign (+), **A**lways **P**ositive (+).

4. **Factoring by Grouping**
   When a polynomial has four terms, you can often factor it by splitting it down the middle into two groups of two.

   $12xy + 3zx + 9zy + 4x^2$

   Group terms that share obvious factors together.

   $\left(12xy + 4x^2\right) + (3zx + 9zy)$

   Pull out GCF from each group:

   $4x(3y + x) + 3z(x + 3y)$

   Rewrite: Notice that $(3y + x)$ and $(x + 3y)$ are exactly the same. Treat this entire parenthesis as a common factor and pull it out:

   $(4x + 3z)(3y + x)$

COMPLEX NUMBERS

Historically, mathematicians ran into a wall when trying to take the square root of a negative number (e.g., $\sqrt{-4}$), since no real number multiplied by itself results in a negative value. To solve this, the imaginary unit i was created.

Definition: $i = \sqrt{-1}$

The Cyclic Powers of i

When you raise i to consecutive integer powers, it forms a repeating pattern or cycle of 4 distinct values ($i, \ -1, \ -i, \ 1$):

- **First power:** $i^1 = i$
- **Second power:** $i^2 = -1$ *(This is a key identity: squaring an imaginary number yields a real negative number)*
- **Third power:** $i^3 = i^2 \cdot i = (-1) \cdot i = -i$
- **Fourth power:** $i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) = 1$

Odd roots of negative numbers are real, not imaginary.

$$(-1)(-1)(-1) = -1$$

The value of $\sqrt[3]{-1}$ is simply the real number -1. Imaginary numbers only occur when taking **even** roots of negative numbers.

RULES OF INEQUALITY

An inequality compares two expressions using relations like less than (<), greater than (>), less than or equal to (≤), or greater than or equal to (≥). Solving them is almost exactly like solving regular equations, with **one golden rule**.

Whenever you multiply or divide both sides of an inequality by a **negative number**, you must **flip the direction** of the inequality sign.

a. **Single Inequality**

$$4 - 3x \leq -5$$

$$-3x \leq -5 - 4$$

$$-3x \leq -9$$

$$x \geq \frac{-9}{-3}$$

$$x \geq 3$$

The inequality is true for any number greater than or equal to 3.

b. Compound Inequality

A compound inequality bounds a variable on two sides simultaneously. Whatever operation you apply, you must perform it on all three parts (left, middle, and right).

$$2 > -3 - 3x \geq -7$$

$$2 + 3 > -3x \geq -7 + 3$$

$$5 > -3x \geq -4$$

$$\frac{5}{-3} < x \leq \frac{-4}{-3}$$

$$-\frac{5}{3} < x \leq \frac{4}{3}$$

The variable x is strictly greater than $-\frac{5}{3}$ and less than or equal to $\frac{4}{3}$.

FUNCTIONS

A function is a machine that takes an input, processes it, and gives back an output. In algebra, we define this machine using strict relational rules.

Relation: Any set of ordered pairs (x,y). It simply maps an input value to an output value.
Function: A special type of relation where **no two distinct pairs have the same first coordinate (x)**. In other words, every single input (x) must point to exactly *one* output (y). If an input points to two different outputs, it is not a function.
Domain: The complete set of all first coordinates (all the possible input x-values) that yield a valid, real output.
Range: The complete set of all second coordinates (all the resulting output y-values).

Consider the finite relation provided:

$$Relation = \{(0, 2),\ (4, 23),\ (90, 35)\}$$

- **Is it a function?** Yes. Look at the inputs (0, 4, 90). None of them repeat, meaning each input has a unique output.
- **Domain:** {0, 4, 90} (the collection of x-coordinates)
- **Range:** {2, 23, 35} (the collection of y-coordinates)

FUNCTIONAL NOTATION AND EVALUATION

We write functions using the notation f(x), read aloud as **"f of x."** Here, x represents the empty placeholder slot where your input goes. To evaluate a function, substitute the value inside the parentheses everywhere you see an x in the equation.

$$f(x) = x^2 + x - 2$$

1. Evaluate f(0) (Numerical Substitution)

Replace every x with 0: $f(0) = 0^2 + 0 - 2 = -2$

2. Evaluate f(a) (Variable Substitution)

Replace every x with the variable a: $f(a) = a^2 + a - 2$

DETERMINANTS

A **matrix** is a rectangular grid of numbers. If a matrix has the exact same number of rows and columns (like a 2 x 2 or 3 x 3), it is called a **square matrix**.

A **determinant** is a single, distinct number calculated from a square matrix. It is denoted by vertical bars around the matrix name, like $|A|$. Determinants are incredibly useful tools used to invert matrices and solve systems of linear equations using methods like **Cramer's Rule**.

In engineering, determinants are widely used to analyze and solve large systems of linear equations that govern physical structures, electrical networks, and fluid dynamics. For a practical example, structural engineers utilize determinants when determining the internal forces acting on a complex bridge truss or building frame. If the determinant equals zero, it reveals an unstable structure that could collapse, allowing engineers to modify the design safely before construction begins.

Evaluating a 3 x 3 Determinant

The image shows a general 3 x 3 matrix A

$$A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

To compute this determinant without a calculator, we use a classical technique called **Sarrus's Rule** (or the diagonal expansion method).

1. **Downward Diagonals (Positive):** Multiply the elements along the three main diagonals stretching from the top-left to the bottom-right, and add them together:

$$Downwards = (a \cdot e \cdot i) + (b \cdot f \cdot g) + (c \cdot d \cdot h)$$

2. **Upward Diagonals (Negative):** Multiply the elements along the three diagonals stretching from the bottom-left to the top-right, and add them together:

$$Upwards = (g \cdot e \cdot c) + (h \cdot f \cdot a) + (i \cdot d \cdot b)$$

3. **Subtract the two groups:**

$$|A| = (aei + bfg + cdh) + (ceg + afh + bdi)$$

The Mechanical Engineering (ME) Board Examination is a race against time, making the advanced scientific calculator (such as the Casio fx-991ES Plus or fx-991EX) an indispensable tool for survival. It acts as a powerful computing partner that drastically reduces manual computation errors and bypasses long, tedious algebraic manipulations. Mastering your calculator allows you to skip manually expanding equations, factoring polynomial systems, or solving complex arithmetic by hand.

Strategic Calculator Techniques for Algebra Foundations

1. Bypassing Factoring and Simplifying with the CALC Feature

When a multiple-choice question asks you to identify the factored form of a long polynomial expression (like the grouping example $\left(12xy + 4x^2\right) + (3zx + 9zy)$, you do not need to factor it manually. You can use the variable storage or **CALC** function to test numerical values.

- **The Technique:** Input the original algebraic expression exactly as written into your calculator using the ALPHA keys to type out the variables (e.g., X, Y, Z). Press the CALC button, and the screen will prompt you to assign temporary numbers to your variables (e.g., let X=2, Y=3, Z=4). Note the resulting number. Next, input the multiple-choice options one by one, hitting CALC with those exact same values. The option that yields the identical numerical result is mathematically equivalent to the original expression.

2. Eliminating Sign Mistakes in Inequalities via TABLE Mode

Inequalities are highly prone to sign-flipping mistakes when handled manually. If you are solving a single or compound inequality, you can use your calculator's **TABLE** function (MODE 7 or MODE 9) to visually verify your boundary conditions instantly.

- **The Technique:** Move all terms to one side so the inequality is compared to zero (e.g., $4 - 3x \leq -5 \Rightarrow 4 - 3x + 5 \leq 0$). Enter the function $f(x) = 4 - 3x + 5$ into the table mode. Set the table parameters to start at a value lower than your suspected boundary and end past it (e.g., Start: 0, End: 5, Step: 1). Scroll through the generated table to see exactly where $f(x)$ changes signs or becomes less than or equal to zero. If the table shows negative outputs starting precisely at X=3 and beyond, you instantly confirm that x ≥ 3 is correct without risking a manual negative-division slip.

3. Immediate Evaluation of Functions and Evaluating Complex Limits

When dealing with functional notation like evaluating $f(a) = a^2 + a - 2$, or testing behaviors near **Indeterminate Forms** (such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$), manual substitution takes too long.

- **The Technique:** For standard functions, simply type $x^2 + x - 2$, press CALC, type your target number, and press =. For indeterminate limit problems where an expression approaches a troublesome value (like X→0 or X→1), you cannot plug the exact number in because it triggers a Math ERROR. Instead, trick the calculator by entering a value infinitely close to the limit. If the limit approaches 3, press CALC and input 3.0000001 or 2.9999999. The calculator will instantly resolve the expression and display the precise finite value the indeterminate form is hiding.

4. Direct Matrix Entry for 3×3 Determinants

Computing a 3×3 determinant by hand using Sarrus's Rule involves six sets of multiplications and multiple sign distributions, leaving it highly vulnerable to basic arithmetic errors under exam pressure.

- **The Technique:** Shift your calculator into Matrix Mode (MODE 6). Choose MatA, select the 3×3 dimension, and fill the grid with your coefficients. Once stored, clear the screen with AC, press SHIFT 4 (Matrix), choose the det command, and feed it your matrix: det(MatA). Pressing = instantly provides the correct distinct scalar value, completely wiping out the risk of a miscalculated diagonal diagonal path.